Topological Properties of Quantum Information Channels (Mathematical Aspects of Quantum Information and Quantum Chaos)

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1 Introduction

It is well known that the theory of classical communication systems are fairly different from that of quantum communication systems. For example, though information sources in the classical theory are formulated as complete event systems and information sources in the quantum theory are formulated as density operators, not all entropy values attached to the density operators are finite but all entropy values attached to the complete event systems are finite. The following table clarifies the difference between the classical information theory and the quantum information theory.

	classical cases	quantum cases
information sources	complete event systems	density operators
information channels	transition matrices	completely positive mappings
entropy	Shannon's entropy	von Neumann's entropy
finite values	all	few
infinite values	none	almost all
dimension	none	Ohya's entropy dimension

In this paper, it is shown that Ohya's entropy dimension on the set of all normal states is a surjective and homeomorphic isomorphism invariant. Moreover, the new topology over the set of all bounded linear mappings defined on the set of all operators with values in the same set is defined. Finally, the Banach-Alaoglu type theorem of the unital completely positive mappings is proved.

2 Ohya's entropy dimension on the set of all normal states

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all positive integers, the set of all real numbers and the set of all complex numbers, respectively. Let $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$ be a separable Hilbert space and the set of all bounded operators on $\mathcal{H}$, respectively. Let $\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}))$ be the set of all normal states on $\mathcal{B}(\mathcal{H})$. If $\mathcal{S}$ is a weakly* compact and convex subset of $\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}))$, then the set of all extremal points belonging to $\mathcal{S}$, which is denoted by $ex\mathcal{S}$, is non-empty. For any normal state $\phi\in\mathcal{S}$, if there exist both a non-negative sequence $\{\lambda_k; k \in \mathbb{N}\}$ satisfying $\sum_k\lambda_k = 1$ and a sequence of normal states $\{\phi_k; k \in \mathbb{N}\} \subset ex\mathcal{S}$, which enable ϕ to be represented by the following countable convex combination:

$$\phi = \sum_{k=1}^{\infty}\lambda_k\phi_k,$$

then, we define $D(\phi, \mathcal{S})$ as the set of all non-negative sequences that enable ϕ to be represented by the above way. Now, for any positive number $\alpha \neq 1$, Ohya's $(\mathcal{S}, \alpha)$-entropy of ϕ is defined as

$$S(\phi, \mathcal{S}, \alpha) = inf\left\{\frac{log\sum_{k=1}^{\infty}\lambda_k^{\alpha}}{1-\alpha}; \{\lambda_k; k \in \mathbb{N}\} \in D(\phi, \mathcal{S})\right\}.$$

Here, Ohya's $\mathcal{S}$-entropy dimension of ϕ is defined as

$$d(\phi, \mathcal{S}) = inf\{\alpha > 0; S(\phi, \mathcal{S}, \alpha) < \infty\}.$$

Throughout this paper, we will treat the case that $\mathcal{S} = \mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}))$ holds and we will abbreviate $d(\phi, \mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H})))$ to $d(\phi)$ for simplicity.

Theorem 2.1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces, and h be a surjective and identity-preserving *-isomorphism on $\mathcal{B}(\mathcal{H}_1)$ with values in $\mathcal{B}(\mathcal{H}_2)$. Then, for any normal state ϕ defined on $\mathcal{H}_2$, $d(\phi) = d(\phi o h)$ holds.

Proof. For any $x \in \mathcal{H}_2$ satisfying $\|x\| = 1$, the vector state constructed from x, which is denoted by ω_x, is defined as

$$\omega_x(A) = < x|A|x >, \quad A \in \mathcal{B}(\mathcal{H}_2).$$

Here, we can assume that ϕ is represented by

$$\begin{aligned} \rho &= \sum_{k=1}^{\infty}\lambda_k|e_k >< e_k|, \\ \phi(A) &= tr(\rho A), \quad A \in \mathcal{B}(\mathcal{H}_2), \end{aligned}$$

where $\{\lambda_k\}$ is a non-negative sequence satisfying $\sum_k \lambda_k = 1$, and $\{e_k\}$ is an orthonormal system of $\mathcal{H}_2$. Then, ϕ can be represented by

$$\phi = \sum_{k=1}^{\infty}\lambda_k\omega_{e_k}.$$

Since $\phi o h = 0$ implies that $\phi = 0$ holds , $j \neq k$ implies that $\omega_{e_j} o h \neq \omega_{e_k} o h$ holds. Therefore, it is sufficient to prove that, for any positive integer k, $\omega_{e_k} o h$ belongs to $ex\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}_1))$ holds. Let ω be an element of $ex\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}_2))$, ψ be $\omega o h$ and $\{\mathcal{H}_\omega, \pi_\omega, x_\omega\}$ (resp. $\{\mathcal{H}_\psi, \pi_\psi, x_\psi\}$) be the cyclic representation of $\mathcal{B}(\mathcal{H}_2)$ (resp. $\mathcal{B}(\mathcal{H}_1)$) constructed from ω (resp. ψ). Let $h_{\omega,\psi}$ be the operator on $\{\pi_\psi(B)x_\psi; B\in\mathcal{B}(\mathcal{H}_1)\}$ with values in $\{\pi_\omega(A)x_\omega; A\in\mathcal{B}(\mathcal{H}_2)\}$ defined as

$$h_{\omega,\psi}\pi_\psi(B)x_\psi = \pi_\omega(h(B))x_\omega, \quad B\in\mathcal{B}(\mathcal{H}_1).$$

Then, for any $B, C \in \mathcal{B}(\mathcal{H}_1)$, we have

$$\begin{aligned}
< h_{\omega,\psi}\pi_\psi(B)x_\psi | h_{\omega,\psi}\pi_\psi(C)x_\psi > &= < \pi_\omega(h(B))x_\omega | \pi_\omega(h(C))x_\omega > \\
&= < x_\omega | \pi_\omega(h(B)^* h(C))x_\omega > \\
&= < x_\omega | \pi_\omega(h(B^*C))x_\omega > \\
&= \omega(h(B^*C)) = \psi(B^*C) \\
&= < x_\psi | \pi_\psi(B^*C)x_\psi > \\
&= < \pi_\psi(B)x_\psi | \pi_\psi(C)x_\psi > .
\end{aligned}$$

These equalities imply that $h^*_{\omega,\psi}h_{\omega,\psi}$ is the identity mapping. It is clear that the uniform closure of $\{\pi_\omega(h(B))x_\omega; B \in \mathcal{B}(\mathcal{H}_1)\}$ is exactly equal to $\mathcal{H}_\omega$, because h is surjective. Therefore, $h_{\omega,\psi}$ can be uniquely extended to an isometry defined on $\mathcal{H}_\psi$. Since, for any $B, C \in \mathcal{B}(\mathcal{H}_1)$, we have

$$\begin{aligned}
h_{\omega,\psi}\pi_\psi(B)h^*_{\omega,\psi}\pi_\omega(h(C))x_\omega &= h_{\omega,\psi}\pi_\psi(B)h^*_{\omega,\psi}h_{\omega,\psi}\pi_\psi(C)x_\psi \\
&= h_{\omega,\psi}\pi_\psi(BC)x_\psi \\
&= \pi_\omega(h(BC))x_\omega \\
&= \pi_\omega(h(B))\pi_\omega(h(C))x_\omega,
\end{aligned}$$

these equalities imply that $h_{\omega,\psi}\pi_\psi(B)h^*_{\omega,\psi} = \pi_\omega(h(B))$ holds for any $B \in \mathcal{B}(\mathcal{H}_1)$, and

$$\{\pi_\psi(B); B \in \mathcal{B}(\mathcal{H}_1)\}' = h^*_{\omega,\psi}\{\pi_\omega(h(B)); B \in \mathcal{B}(\mathcal{H}_2)\}' h_{\omega,\psi} = \mathbb{C}I,$$

where I means the identity mapping on $\mathcal{H}_\psi$, and $\mathcal{A}'$ means the commutant of an algebra $\mathcal{A}$. These equalities imply that the cyclic representation $\{\mathcal{H}_\psi, \pi_\psi, x_\psi\}$ is irreducible, therefore, we obtain the conclusion. □

3 The Banach-Alaoglu type theorem of the unital and completely positive mappings on $\mathcal{B}(\mathcal{H})$

Let $\mathcal{T}(\mathcal{H})$ be the set or all operators of trace class and $\mathcal{L}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ be the set of all bounded linear transformations defined on $\mathcal{B}(\mathcal{H})$ with values in $\mathcal{B}(\mathcal{H})$. It follows from Banach-Alaoglu's theorem that the closed unit ball of $\mathcal{B}(\mathcal{H})$, which is denoted by $\mathcal{U}(\mathcal{H})$, is a weakly* compact subset of $\mathcal{B}(\mathcal{H})$ because the conjugate space of $\mathcal{T}(\mathcal{H})$ is exactly equal to $\mathcal{B}(\mathcal{H})$. Therefore, the topological space $\mathcal{U}(\mathcal{H})^{\mathcal{U}(\mathcal{H})}$ which is equipped with Tychonoff's product topology is a compact Hausdorff space.
Here, we define the transformation topology, which is denoted by τ_T, as the locally convex topology over $\mathcal{L}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ determined by the family of the following semi-norms:

$$\Lambda(\cdot) \mapsto |trace[\sigma\Lambda(A)]|,$$

where σ runs over all elements of $T(\mathcal{H})$ and A runs over all elements of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ be the set of all unital and completely positive mappings defined on $\mathcal{B}(\mathcal{H})$ with values in $\mathcal{B}(\mathcal{H})$. Then we have now the following theorem

Theorem 3.1. $\mathcal{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ is a compact subset of the topological space $(\mathcal{L}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})), \tau_T)$.

Proof. It is clear that, for any $\Lambda \in \mathcal{H}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$, the equalities

$$\|\Lambda\| = \|\Lambda(I)\| = 1$$

hold. If we assume that $\Pi \in \mathcal{L}(\mathcal{B}(calH), \mathcal{L}(\mathcal{H}))$, $\{\Lambda_j\} \subset \mathcal{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ $\Lambda \to \Pi$ holds, then, for any $n \in \mathbb{N}$, for any $x \in \mathcal{H}$ and for any $\{A_k; 1 \leq k \leq n\}, \{B_k; 1 \leq k \leq n\} \subset \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned}
0 &\leq \; < x | \sum_{k,l=1}^{n} B_k^* \Lambda_j (A_k^* A_l) B_l | x > \\
&= trace \left[\sum_{k,l=1}^{n} |B_k x >< B_l x| \Lambda_j (A_k^* A_l) \right] \\
&= \sum_{k,l=1}^{n} trace \left[|B_k x >< B_l x| \Lambda_j (A_k^* A_l) \right] \\
&\to \sum_{k,l=1}^{n} trace \left[|B_k x >< B_l x| \Pi (A_k^* A_l) \right] \\
&= \; < x | \sum_{k,l=1}^{n} B_k^* \Pi (A_k^* A_l) B_l | x > .
\end{aligned}$$

References

[1] S. Akashi, Superposition representability problems of quantum information channels, Open Systems and Information Dynamics, 4(1997), 45-52.

[2] S. Akashi, Quantum entropy theoretic aspects of the 13th problem formulated by Hilbert, Journal of the Ramanujan Mathematical Society, 12(1997), 135-146.

[3] R. S. Ingarden and K. Urbanik, Quantum information thermodynamics, Acta Phys. Polon., 21(1962), 281-304.

[4] R. S. Ingarden, A. Kossakowski and M. Ohya, Information Dynamics and Open Systems, Kluwer Academic Publishers, Dordrecht, 1997.

[5] M. Ohya, Some aspects of quantum information theory and their applications to irreversible processes, Rep. Math. Phys., 27(1989), 19-47.